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Kac–Moody Algebra, Nonlocal Symmetries, and Backlund Transformation for KdV Equation

A. Roy Chowdhury[1] and Swapna Roy[1]

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Gauge transformation of the Lax eigenfunction through the explicit use of Lie group generators is seen to generate a two-parameter Backlund transformation. Explicit integration of this in two particular cases leads to $\operatorname{sech}^2\theta$ type and rational solutions starting from the trivial one. A method is indicated to generate infinitesimal transformations around u in the sense of Steudel, which in this case leads to a nonlocal structure of transformations.

Lie algebra has been successfully used in the study of nonlinear integrable systems over the last two decades (Olshanetsky and Perelomov, 1981; Kupershmidt, 1987). Later, affine Kac–Moody algebras were also incorporated and elegant results were obtained by various authors (Kupershmidt and Wilson, 1985; Guil, 1984, 1985; Szmiglielski, 1988). Among the most important results are deformation, the Miura map, and the Backlund transformation, deduced via the automorphism of the Lie algebra. There has also been considerable study on the gauge transformation theory for nonlinear systems (Eichenherr and Honnerkamp, 1981; Honnerkamp, 1981). Here we show that it is possible to deduce the Backlund transformation for the KdV problem with an explicit realization of the gauge transformation of the Lax eigenfunction in a particular Lie algebra.

To start with, let us consider the prolongation structure associated with the KdV equation (Van Eck, 1983):

$$u_t + 12uu_x + u_{xxx} = 0 \tag{1}$$

written as

$$w = dy + A\,dx + B\,dt$$

[1]High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta-700032, India.

with A, B given in the form

$$A = y - (2u + T)z$$
$$B = (2u_{xx} + 8u^2 - 4T^2 - 4uT)z + (4T - 4u)y - 2u_x h \tag{2}$$

where (h, y, z) span the Lie algebra and satisfy

$$[h, y] = 2y; \qquad [h, z] = -2z; \qquad [y, z] = h \tag{3}$$

T is an arbitrary parameter involved in the definition of the infinite-dimensional algebra $A_1 \otimes C(T)$. $C(T)$ is the ring of polynomials in T needed to close the prolongation structure.

The differential form w is equivalent to the Lax pair

$$\psi_x = A\psi, \qquad \psi_t = B\psi \tag{4}$$

Let us transform ψ as

$$\psi' = g\psi \tag{5}$$

where $g \in A_1$ and can be parametrized as

$$g = e^{\alpha h} e^{\beta y} e^{\gamma z} \tag{6}$$

α, β, γ are arbitrary functions of (x, t). If we demand equation (4) to be form invariant, then we must have

$$\psi'_x = A'\psi', \qquad \psi'_t = B'\psi' \tag{7}$$

Using (5) in (6), we get

$$\begin{aligned} A' &= g_x g^{-1} + gAg^{-1} \\ B' &= g_t g^{-1} + gBg^{-1} \end{aligned} \tag{8}$$

using now g as given in (6), we get

$$\begin{aligned} \alpha_x &= -\gamma - \beta(2u + T)e^{2\alpha} \\ \gamma_x &= -(2u' + T) + (2u + T)e^{2\alpha} - \gamma^2 \\ \beta_x &= 1 - e^{-2\alpha} + \beta^2(2u + T)e^{2\alpha} + 2\beta\gamma \end{aligned} \tag{9}$$

where u' is the new nonlinear field corresponding to (A', B'). Similarly, from the time part we get

$$\begin{aligned} \alpha_t &= 2(u_x - 4u') + \beta(2u_{xx} + 8u^2 - 4T^2 - 4uT)e^{2\alpha} - (4T - 4u')\gamma \\ \beta_t &= (4T - 4u')(1 - \beta\gamma) - \beta^2(2u_{xx} + 8u^2 - 4T^2 - 4uT)e^{2\alpha} \\ &\quad - (4T - 4u)e^{-2\alpha} + 4u'_x\beta \\ \gamma_t &= (2u'_{xx} + 8u'^2 - uT^2 - 4u'T) - e^{2\alpha}(2u_{xx} + 8u^2 - 4T^2 - 4uT) \\ &\quad - 4u'_x\gamma - \gamma^2(4T - 4u') \end{aligned} \tag{10}$$

We now invoke the condition that it is possible to generate u from u' via g^{-1} and g^{-1} is obtained from g by reversing the sign of (α, β, γ). This immediately leads to

$$e^{2\alpha} = \left(\frac{2u'+T}{2u+T}\right)^{1/2}$$
$$\gamma = \int (u'-u)\, dx + g(t) \tag{11}$$

Using (11) and (9), we get

$$\left[\int (u-u')\, dx + g(t)\right]^2 = -u' - u - T + (2u+T)^{1/2}(2u'+T)^{1/2} \tag{12}$$

It is interesting to observe that equation (12) connects the solutions (u, u') and implies $u = u'$ if we set $g = T = 0$. So we may say that (12) is the space part of the required BT. On the other hand, using equation (10), we get the time part of the BT as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\int (u-u')\, dx + g(t)\right\} \\
&\quad = (2u'_{xx} + 8u'^2 - 4T^2 - 4u'T) \\
&\qquad - (2u_{xx} + 8u^2 - 4T^2 - 4uT)\left(\frac{2u'+T}{2u+T}\right)^{1/2} \\
&\qquad - 2u'_x\gamma - \gamma^2(4T - 4u')
\end{aligned} \tag{13}$$

To check that equation (12) is actually a BT, we set $g = 0$, $u' = 0$, and $v^2 = 2u + T$; then (12) reduces to a simple ordinary differential equation for v,

$$\frac{dv}{dx} = \frac{1}{\sqrt{2}}(v^2 - T)$$

which immediately integrates to

$$v = -T^{1/2} \tanh \frac{\alpha x + h'}{2} \tag{14}$$

if $T = \alpha^2/2$, so that

$$u = \frac{T}{2}\operatorname{sech}^2 \frac{\alpha x + h'}{2} = \frac{\alpha^2}{4}\operatorname{sech}^2 \frac{\alpha x + h'}{2} \tag{15}$$

On the other hand, we may also set $T = 0$, $u' = 0$, and $u = v_x$, leading to

$$v_{xx} + 2vv_x + 2g(t)v_x = 0$$

which upon integration leads to

$$v_x + v^2 + 2gv = C \tag{16}$$

and one immediately ends up with a solution

$$v = \beta \tanh(\beta x) - g$$

So,

$$\begin{aligned} u &= v_x = \beta^2 \operatorname{sech}^2(\beta x) \\ \beta &= C + g^2(t) \end{aligned} \tag{17}$$

On the other hand, if we choose C, the constant of integration, in such a manner that $\beta = 0$, then we get from (15)

$$v_x = -(v+g)^2$$

whence

$$u = -\frac{1}{(x-d)^2} \tag{18}$$

We immediately observe that (15), (17), and (18) are nothing but stationary forms of the solitary and rational type solutions of equation (1).

Lastly we analyze the structure of infinitesimal transformations suggested by the BT (12). Let us again assume $g = 0$ and set

$$u' = u + \sum_{n=1}^{\infty} T^n G_n \tag{19}$$

Then from (12) we get a recursion relation for the G_i by equating various powers of T^i. We can write this as

$$4u \sum \int G_n \int G_{n-k} + 2 \sum \int G_n \int G_{n-k-1} = \sum G_n G_{n-k} \tag{20}$$

We now set $n = 0$, 1, 2 and solve (17), which yields

$$G_0 = 2i\sqrt{u} \exp\left(2i \int \sqrt{u}\right)$$

$$G_1 = \left[\exp\left(2i \int \sqrt{u}\right)\right]\left(2i\sqrt{u} \int \frac{i}{2\sqrt{u}} + \frac{i}{2\sqrt{u}} + 2i\sqrt{u}\right)$$

$$\begin{aligned} G_2 = \left[\exp\left(2i \int \sqrt{u}\right)\right]\Bigg(& -\frac{i}{2}\sqrt{u} \int \frac{1}{\sqrt{u}} \int \frac{1}{\sqrt{u}} - 2\sqrt{u} \int \frac{1}{\sqrt{u}} + \frac{\sqrt{u}}{8} \int \frac{1}{u\sqrt{u}} \\ & + 2i\sqrt{u} - \frac{1}{4u} \int \frac{1}{\sqrt{u}} + \frac{i}{2\sqrt{u}} - \frac{i}{16u\sqrt{u}}\Bigg) \end{aligned}$$

and so on.

Now, as observed by Steudel, if we denote (12) as

$$u' = B_T u$$

where B stands for a BT with a parameter T, and u'' is another solution obtained via $T+\varepsilon$,

$$u'' = B_T + \varepsilon u$$

then we can write

$$u'' = B_T + \varepsilon (B_T)^{-1} u'$$

If we now expand $B_T + \varepsilon (B_T)^{-1}$ in ε, we get

$$u'' = \varepsilon \eta + u'; \qquad \eta = \frac{d}{d\varepsilon} (B_T + \varepsilon B_T^{-1}) \tag{21}$$

and these coefficients η will be the Lie–Backlund symmetries (Van Eck, 1983). Since our initial expansion for the B_T involves nonlocal terms, it is quite simple to ascertain that those in (18) will also be nonlocal.

Lastly, we mention that in this particular situation it is possible to utilize another form of lie group generators,

$$\begin{aligned} g &= (1 + m^+ \cdot y)(1 + n^- \cdot z) e^{\phi^+ h} \\ &= (1 + m^- \cdot z)(1 + n^+ \cdot y) e^{\phi^- h} \end{aligned}$$

with

$$e^{\phi^- - \phi^+} = (1 - m^- m^+)^{-1} y^2 = z^2 = 0$$

and

$$n^\pm = \frac{m^\pm}{1 - m^- m^+}, \qquad h^2 = 1$$

Proceeding along the same lines as before, we deduce

$$\begin{aligned} &u_{xx}[4(u-u') + T]\left[\exp\left(4 \int u_x \, dt\right)\right] \\ &\times \left[\int dt \, 4(u-u') \exp\left(4 \int u_x \, dt\right) + D\right] \\ &- [2u'(u'_x - u_x) + T(u'_x - u_x)] = 0 \end{aligned}$$

which is another form of the x part of the BT and it is not difficult to deduce also a time part via the same route.

The important part of our group-theoretic approach is that it leads to a two-parameter BT, while in the usual BT of the KdV equation usually we have only one constant. In our case one of the parameters is supplied by the Kac–Moody algebra itself and the other one through the constant of integration.

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